Virtual Humans – Winter 23/24

Lecture 3_2 – Procrustes Alignment

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Goal: learn a model of pose and shape

\[ M(0, X_{\text{shape}}) \quad M(X_{\text{pose}}, 0) \quad M(X_{\text{pose}}, X_{\text{shape}}) \]

\[ R \cdot M(X_{\text{pose}}, X_{\text{shape}}) \quad M(X_{\text{pose}}, X_{\text{shape}}) \quad Y \]

\[ X = \{ X_{\text{pose}}, X_{\text{shape}} \} \]
Scan a Lot of People
In lots of poses
To learn a model we need correspondence
Non-rigid Articulated Registration

How?
Today: Rigid Alignment (Procrustes)

\[ \arg\min_S \| (sR \cdot X^T + t) - X'^T \| \]
Surface representation: Mesh

\[ V \equiv \begin{cases} \mathbf{F} &\in \mathbb{N}^{M \times 3} \\ \mathbf{X} &\in \mathbb{R}^{N \times 3} \\ \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \end{cases} \]
Surface representation: Mesh

\[ \nu \equiv \begin{cases} 
F & \in \mathbb{N}^{M \times 3} \\
X & \in \mathbb{R}^{N \times 3} 
\end{cases} \\
\nu \equiv \mathbb{R}^2 \to \mathbb{R}^3 \]
Surface representation: Mesh

How can we rigidly transform a set of points?

• Translate it
  \[ X' = X + t, \quad t \in \mathbb{R}^3 \]

• Rotate it
  \[ X'^T = R \cdot X^T, \quad R \in \text{SO}(3) \]

• Today we’ll consider also

• Scale it
  \[ X' = sX, \quad s \in \mathbb{R} \]
Rigid transformation + scale

\[ R : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]

\[ x' = R x \]
Rigid transformation + scale

\[ R : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]

\[ x' = sRx \]
Rigid transformation + scale

\[ R : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]

\[ x' = sR x + t \]
Rigid transformation + scale

\[ R : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]
\[ x' = sR \cdot x + t \]

\[
\begin{bmatrix}
  x' \\
  1
\end{bmatrix} =
\begin{bmatrix}
  sR & t \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  1
\end{bmatrix}
\]
Procrustes alignment problem definition

\[ X_0 = X + t, \quad t \in \mathbb{R}^3 \]

Diagram:

- \( X \) -> \( S \) -> \( X' \)
Procrustes alignment problem definition

\[ X' = S(X) = (sR \cdot X^T + t)^T \]
Procrustes alignment problem definition

\[ \arg\min_{s, R, t} \| (sRX^T + t) - X'^T \|_F^2 \]
The name Procrustes (Greek: Προκρούστης) refers to a bandit from Greek mythology who made his victims fit his bed either by stretching their limbs or cutting them off.

Procrustes means “he who stretches”
Optimisation Problem

\[ s, R, t = \arg \min_{s, R, t} \sum_{i} \| sRx_i + t - y_i \|^2 \]
Quick recap of SVD

\[ A = U \Sigma V^T \]
Quick recap of SVD

in general, applied to a real matrix:

\[ A = U \Sigma V^T \]

A \equiv (M \times N) \text{ real}

U \equiv (M \times M) \text{ orthogonal, unit norm}

V \equiv (N \times N) \text{ orthogonal, unit norm}

\[ \Sigma \equiv (M \times N) \text{ diagonal} \]

warning: this is not the vertex matrix!
Procrustes alignment steps

\[ X = [x_1 \ldots x_n]^T \quad Y = [y_1 \ldots y_n]^T \]

Matrices of points \( x_i, y_i \in \mathbb{R}^{3 \times 1} \)

\[
\bar{Y}^T X = U \Sigma V^T \\
R = UV^T
\]

Optimal rotation obtained by computing SVD on the point cross-covariance

\[ t = sR\bar{x} - \bar{y} \]

Translation is the centroid difference

\[ s = \frac{\text{tr}(\Sigma)}{\|\bar{X}\|^2_F} \]

Scale is a quotient of eigenvalue sums

* See also clarification slide 1 at the end of slide deck
Proof

\[
s, R, t = \arg \min_{s, R, t} \sum_i \left\| sR \mathbf{x}_i + t - \mathbf{y}_i \right\|^2
\]
Procrustes derivation: translation

\( \mathbf{X}, \mathbf{Y} \in \mathbb{R}^{(N \times 3)} \)

\( \mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^{(3 \times 1)} \)

\( s, \mathbf{R}, t = \arg \min_{s, \mathbf{R}, t} E \)

\[
E \equiv \sum_i \| s \mathbf{R} \mathbf{x}_i + t - \mathbf{y}_i \|^2
\]

Minimize the L2 distance between transformed source points and target points

\[
= \sum_i (s \mathbf{R} \mathbf{x}_i + t - \mathbf{y}_i)^T (s \mathbf{R} \mathbf{x}_i + t - \mathbf{y}_i)
\]

\[
= \sum_i s^2 \mathbf{x}_i^T \mathbf{x}_i + t^T t + \mathbf{y}_i^T \mathbf{y}_i + 2s \mathbf{x}_i^T \mathbf{R}^T t - 2s \mathbf{x}_i^T \mathbf{R}^T \mathbf{y}_i - 2t^T \mathbf{y}_i
\]
Procrustes derivation: translation

We remove the elements that do not depend on the translation and solve for $t$.

$$
t = \arg \min_t E = \arg \min_t \sum_i s^2 x_i^T x_i + t^T t + y_i^T y_i + 2sx_i^T R^T t - 2sx_i^T R^T y_i - 2t^T y_i
$$

$$
= \arg \min_t \sum_i t^T t + 2sx_i^T R^T t - 2t^T y_i
$$

$$
\bar{x} \equiv \frac{\sum_i x_i}{N}, \bar{y} \equiv \frac{\sum_i y_i}{N}
$$

$$
t = \arg \min_t E = \arg \min_t (t^T (2sR\bar{x} + t - 2\bar{y})) = \bar{y} - sR\bar{x}
$$

Compute the centroid of the point clouds

So given $s$ and $R$, we can compute the translation $t$.
Procrustes derivation: Rotation

Subtract the centroid from the points to obtain a simpler expression for $E$

$$
\bar{x}_i \equiv x_i - \bar{x}, \quad \bar{y}_i \equiv y_i - \bar{y}
$$

$$
E = \sum_i \|sRx_i + t - y_i\|^2 = \sum_i \|sR\bar{x}_i - \bar{y}_i\|^2
$$

subject to $R \in SO(3)$

Optimal rotation does not depend on scale

$$
R = \arg \min_R \|R\bar{X}^T - \bar{Y}^T\|_F
$$
Procrustes derivation: Rotation

Subtract the centroid from the points to obtain a simpler expression for $E$

$$\bar{x}_i \equiv x_i - \bar{x}, \bar{y}_i \equiv y_i - \bar{y}$$

$$E = \sum_i \|sR x_i + t - y_i\|^2 = \sum_i \|sR \bar{x}_i - \bar{y}_i\|^2$$

$$R = \arg \min_R \|R \bar{X}^T - \bar{Y}^T\|_F$$

$$= \arg \min_R \langle R \bar{X}^T - \bar{Y}^T, R \bar{X}^T - \bar{Y}^T \rangle_F$$

$$= \arg \min_R \|R \bar{X}^T\|_F^2 + \|\bar{Y}^T\|_F^2 - 2\langle R \bar{X}^T, \bar{Y}^T \rangle_F$$

$$= \arg \max_R \langle R, \bar{Y}^T \bar{X} \rangle_F$$

Optimal rotation does not depend on scale

Rotation does not change norm!

Inner product should be maximum!
Procrustes derivation: Rotation

Subtract the centroid from the points to obtain a simpler expression for $E$

$$
\bar{x}_i \equiv x_i - \bar{x}, \bar{y}_i \equiv y_i - \bar{y}
$$

$$
E = \sum_i \|sR\bar{x}_i + t - y_i\|^2 = \sum_i \|sR\bar{x}_i - \bar{y}_i\|^2
$$

$$
R = \arg \min_R \|R\bar{X}^T - \bar{Y}^T\|_F^2
$$

Optimal rotation does not depend on scale

Rotation does not change norm!

Inner product should be maximum!

Cross-covariance matrix

SVD decomposition

$$
R = \arg \min_R \|R\bar{X}^T - \bar{Y}^T, R\bar{X}^T - \bar{Y}^T\|_F
$$

$$
= \arg \min_R \|R\bar{X}^T\|_F^2 + \|\bar{Y}^T\|_F^2 - 2\langle R\bar{X}^T, \bar{Y}^T\rangle_F
$$

$$
= \arg \max_R \langle R, \bar{Y}^T \bar{X} \rangle_F
$$

$$
= \arg \max_R \langle R, U\Sigma V^T \rangle_F
$$

$$
= \arg \max_R \langle U^T RV, \Sigma \rangle_F
$$

$$
= \arg \max_R \langle S, \Sigma \rangle_F \quad \text{where} \quad S = U^T RV
$$

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Procrustes derivation: Rotation

\[
\arg \max_{R} \langle S, \Sigma \rangle_F \quad \text{where} \quad S = U^T R V
\]

What kind of matrix is \( S \)?

What kind of matrix is \( \Sigma \)?
Procrustes derivation: Rotation

\[
= \arg \max_{\mathbf{R}} \langle \mathbf{S}, \mathbf{\Sigma} \rangle_F \quad \text{where} \quad \mathbf{S} = \mathbf{U}^T \mathbf{R} \mathbf{V}
\]

What kind of matrix is \( \mathbf{S} \)?  
Orthogonal

What kind of matrix is \( \mathbf{\Sigma} \)?  
Diagonal

Hence the quantity above is maximised when \( \mathbf{S} \) equals the identity, hence:

\[
\mathbf{I} = \mathbf{U}^T \mathbf{R} \mathbf{V} \\
\mathbf{R} = \mathbf{U} \mathbf{V}^T
\]

SVD of cross-covariance of pointclouds

\[
\bar{\mathbf{Y}}^T \mathbf{X} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T
\]
Procrustes derivation: scale

Optimize scale given the rotation

\[ s = \arg \min_s E = \arg \min_s \left( \sum_i s^2 \bar{x}_i^T \bar{x}_i + \bar{y}_i^T \bar{y}_i - 2s \bar{x}_i^T R^T \bar{y}_i \right) \]

\[ = \arg \min_s \left( s^2 \sum_i (\bar{x}_i^T \bar{x}_i) - 2s \sum_i (\bar{x}_i^T R^T \bar{y}_i) \right) \]

\[ = \arg \min_s (s^2 a - 2sb) = \frac{b}{a} = \frac{\sum_i (\bar{x}_i^T R^T \bar{y}_i)}{\sum_i (\bar{x}_i^T \bar{x}_i)} \]

\[ = \frac{\text{tr}(X R^T Y^T)}{\|X\|_F^2} = \frac{\text{tr}(R^T Y^T X)}{\|X\|_F^2} = \frac{\text{tr}(V U^T U \Sigma V^T)}{\|X\|_F^2} = \frac{\text{tr}(\Sigma)}{\|X\|_F^2} \]

We used:
1) Trace invariance with shifts
2) \( \bar{Y}^T \bar{X} = U \Sigma V^T, \quad R = UV^T \)
3) Trace equals sum of eigenvals for square matrices
Clarification slides
(not included in the video recordings)
Clarification slide 1 (not in the video)

\[ \bar{X} = [x_1 - \bar{x}, \ldots, x_n - \bar{x}]^T \quad \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \]

Normalized pointcloud. The centroid is subtracted to all points to center the pointcloud at the origin.

Centroid