# Virtual Humans – Winter 23/24

Lecture 3_2 – Procrustes Alignment

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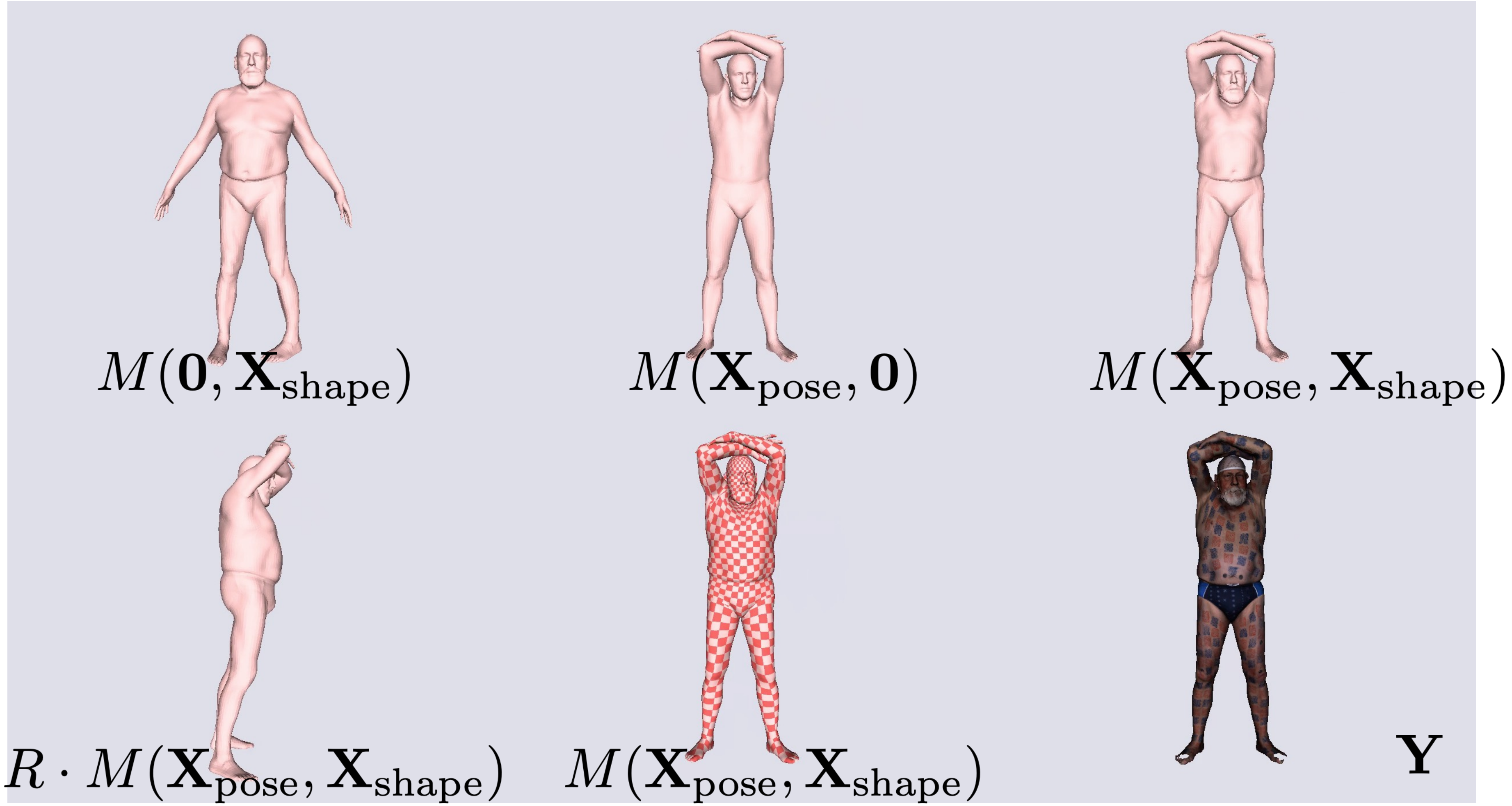
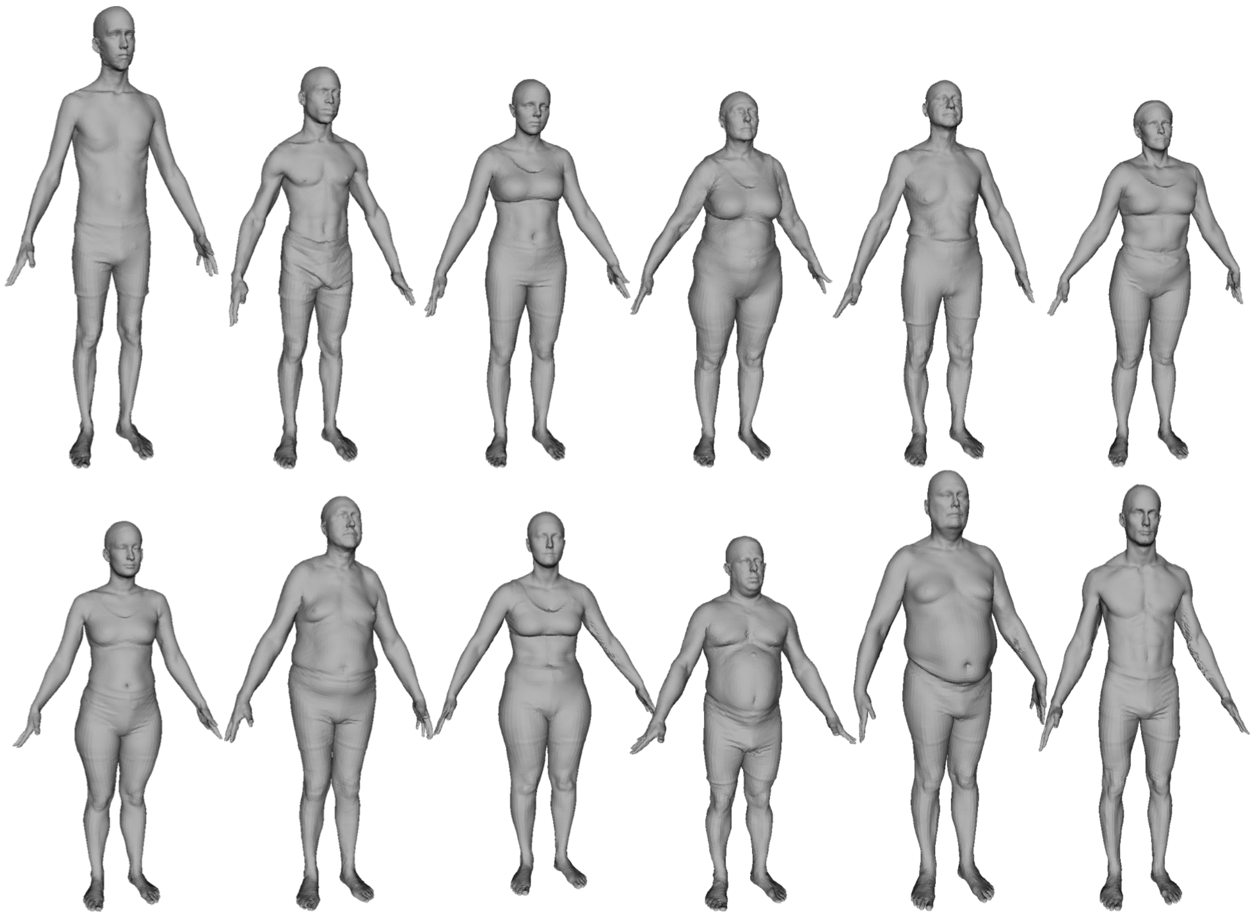
# Goal: learn a model of pose and shape



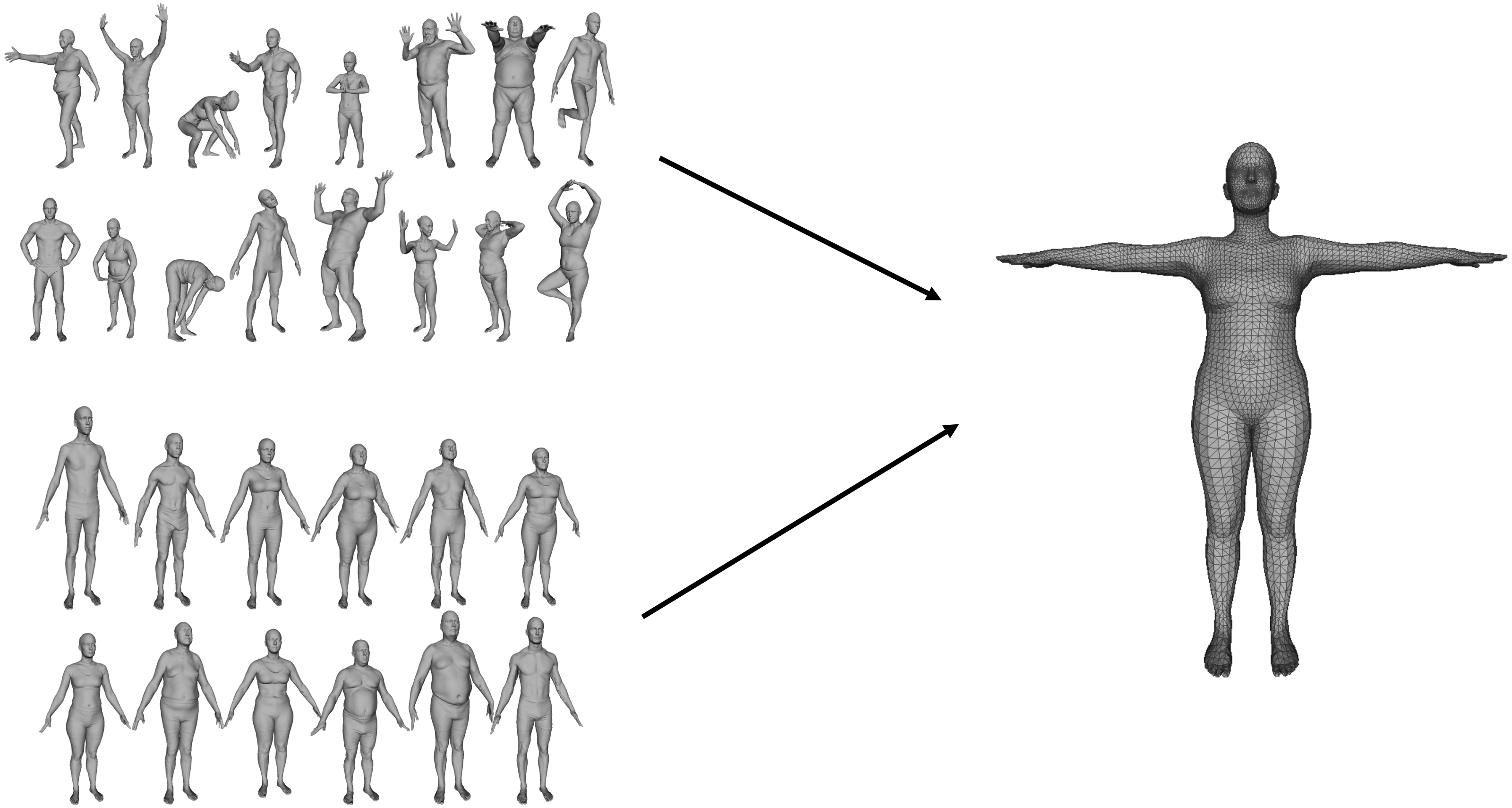

$$\mathbf{X} = \{\mathbf{X}_{\text{pose}}, \mathbf{X}_{\text{shape}}\}$$

# Scan a Lot of People

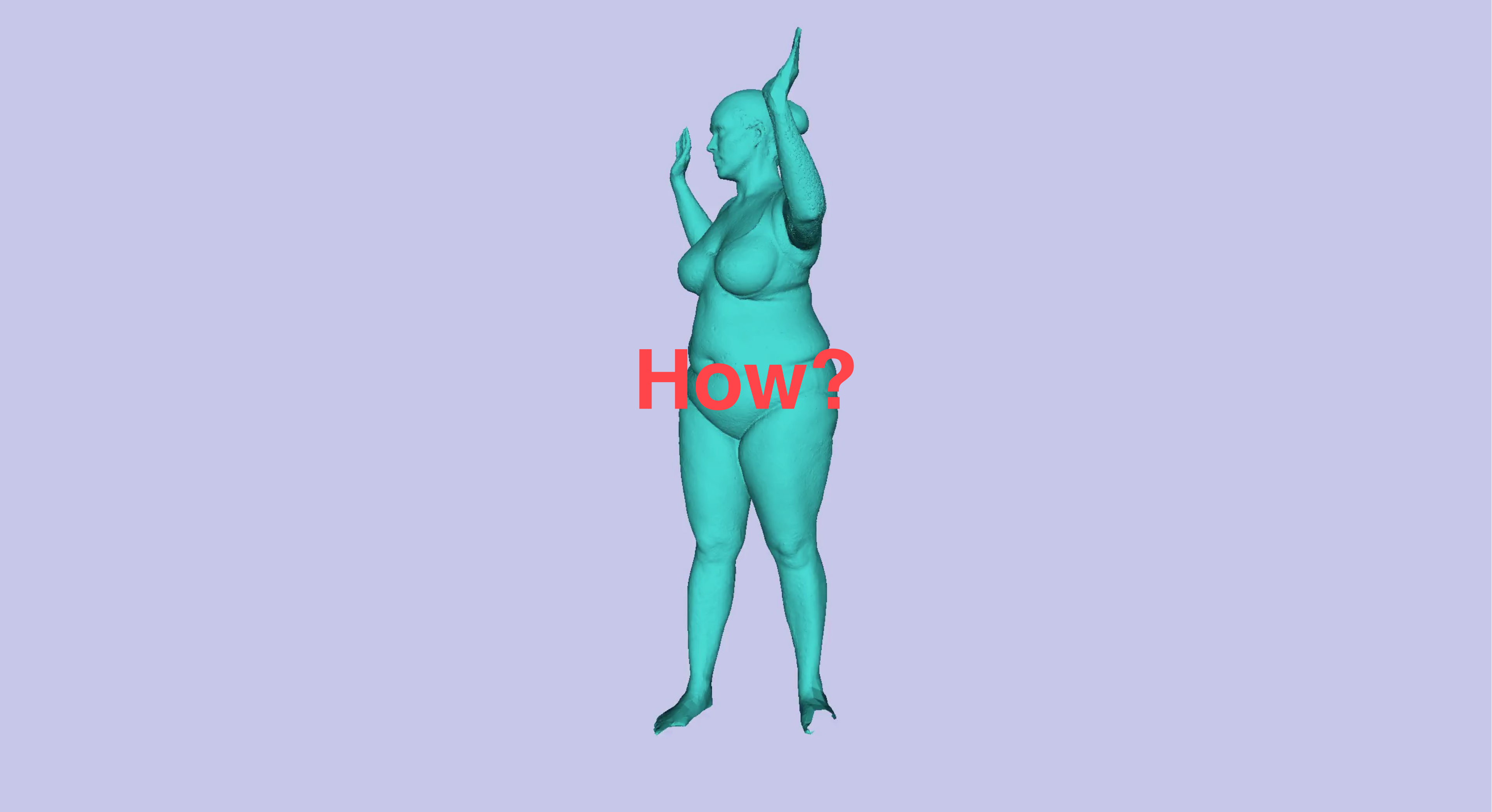


# In lots of poses

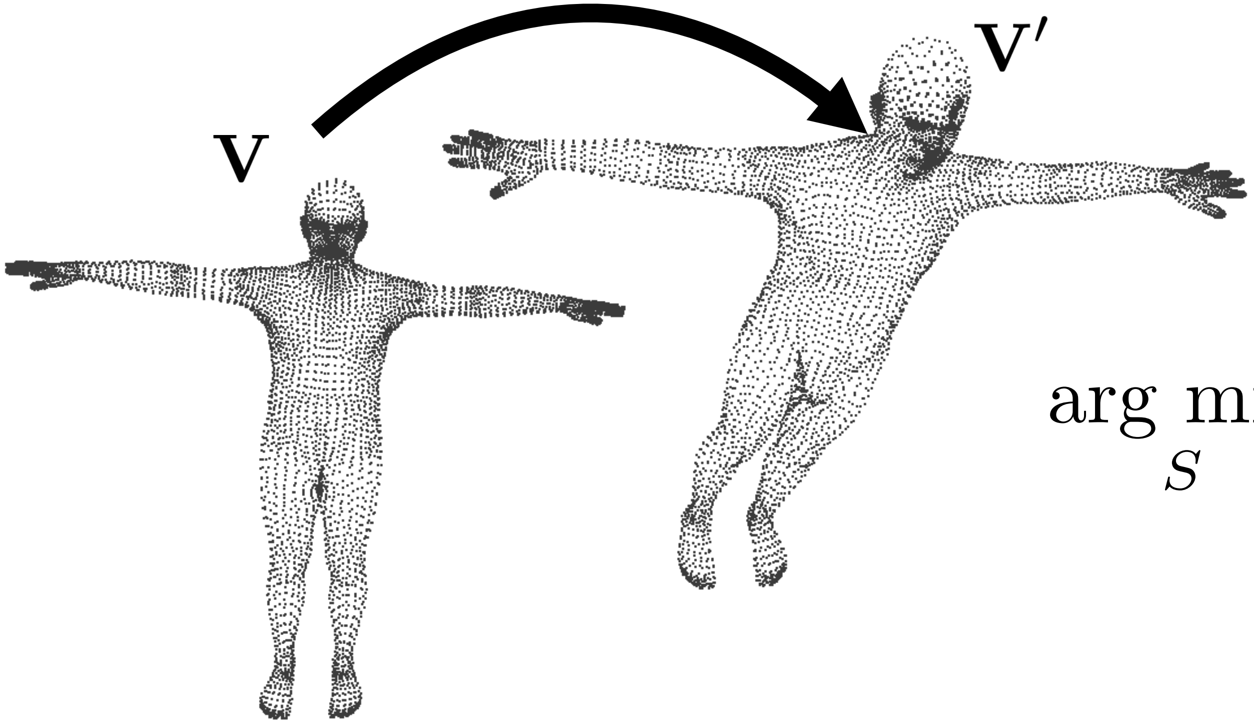
# To learn a model we need correspondence



# Non-rigid Articulated Registration



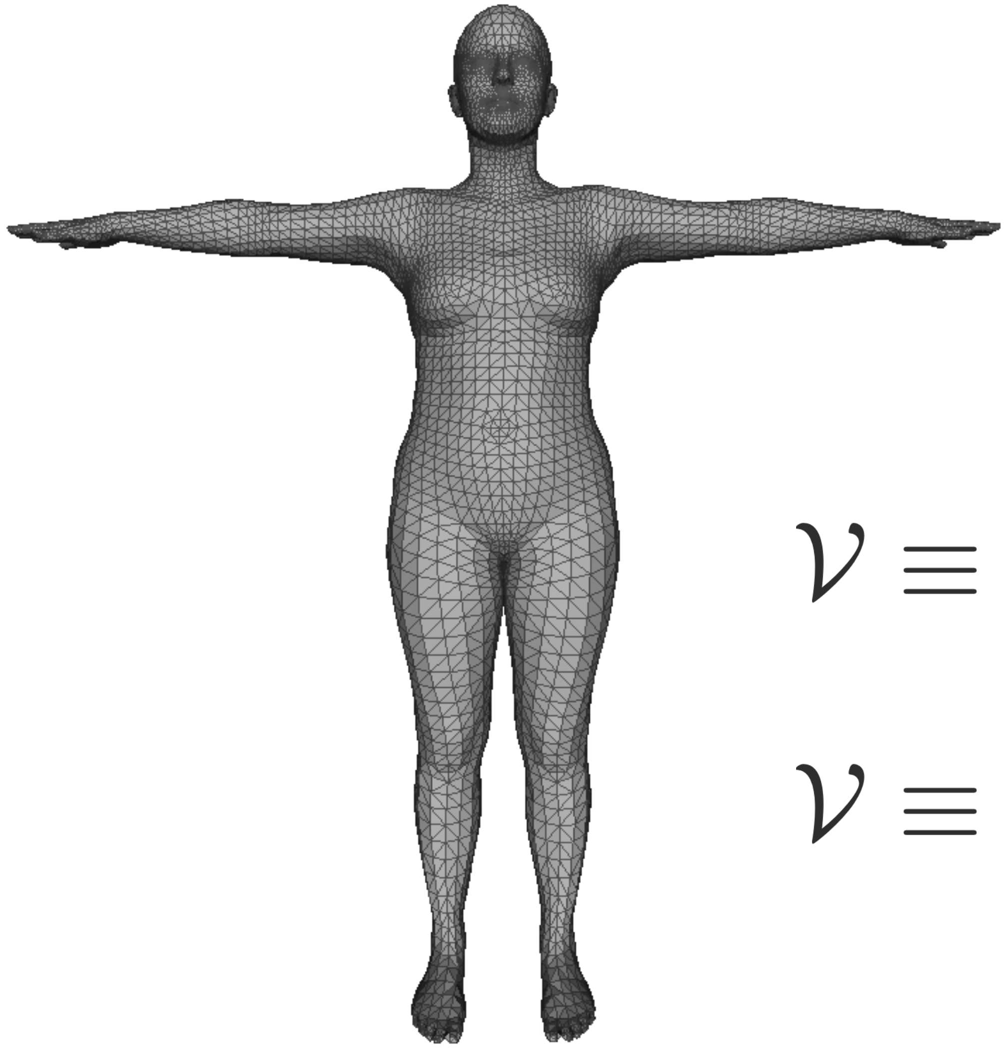
# Today: Rigid Alignment (Procrustes)




$$\arg\min_{S} ||(s\mathbf{R} \cdot \mathbf{X}^T + \mathbf{t}) - \mathbf{X}'^T||$$

?

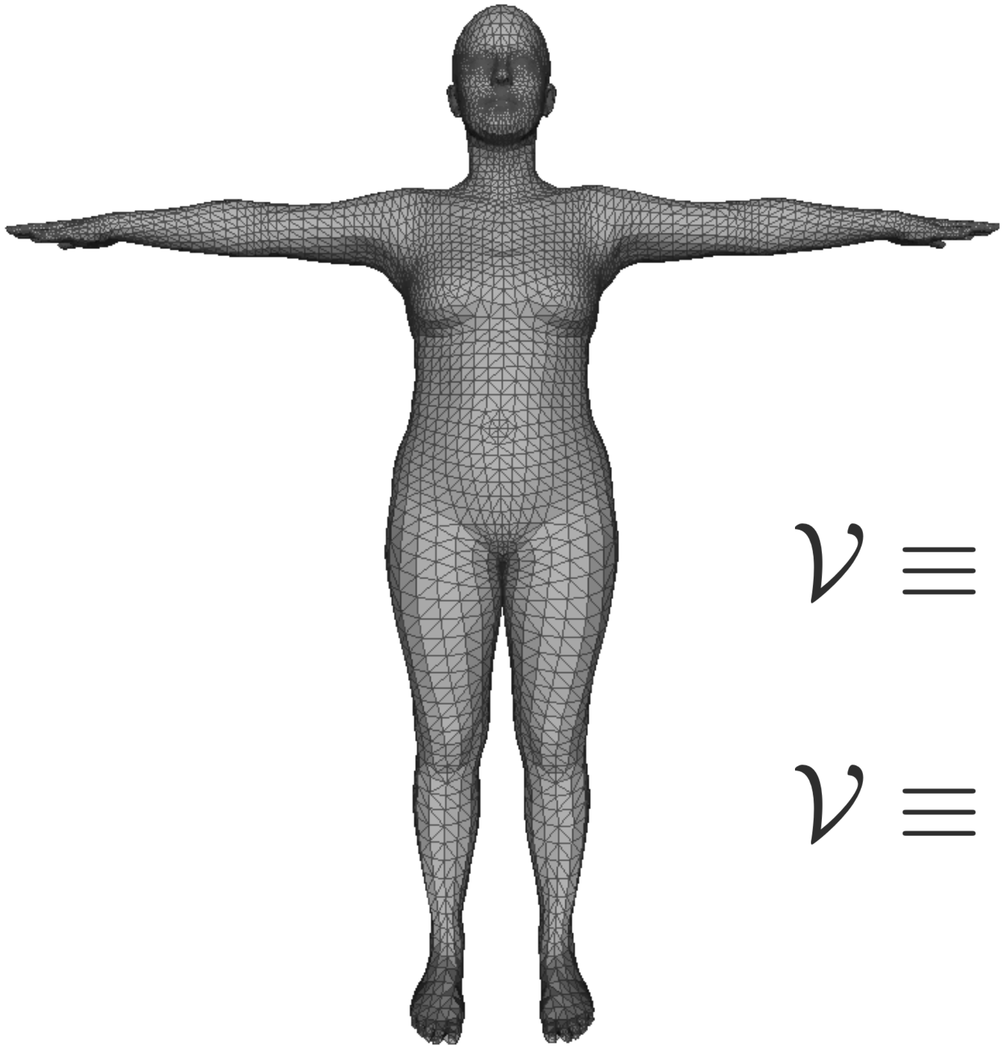
# Surface representation: Mesh



$$\mathcal{V} \equiv \begin{cases} \mathbf{F} \in \mathbb{N}^{M \times 3} \\ \mathbf{X} \in \mathbb{R}^{N \times 3} \end{cases}$$

$$\mathcal{V} \equiv \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

# Surface representation: Mesh



$$\mathcal{V} \equiv \begin{cases} \mathbf{F} \in \mathbb{N}^{M \times 3} \\ \mathbf{X} \in \mathbb{R}^{N \times 3} \end{cases}$$

$$\mathcal{V} \equiv \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

# Surface representation: Mesh

How can we rigidly transform a set of points ?

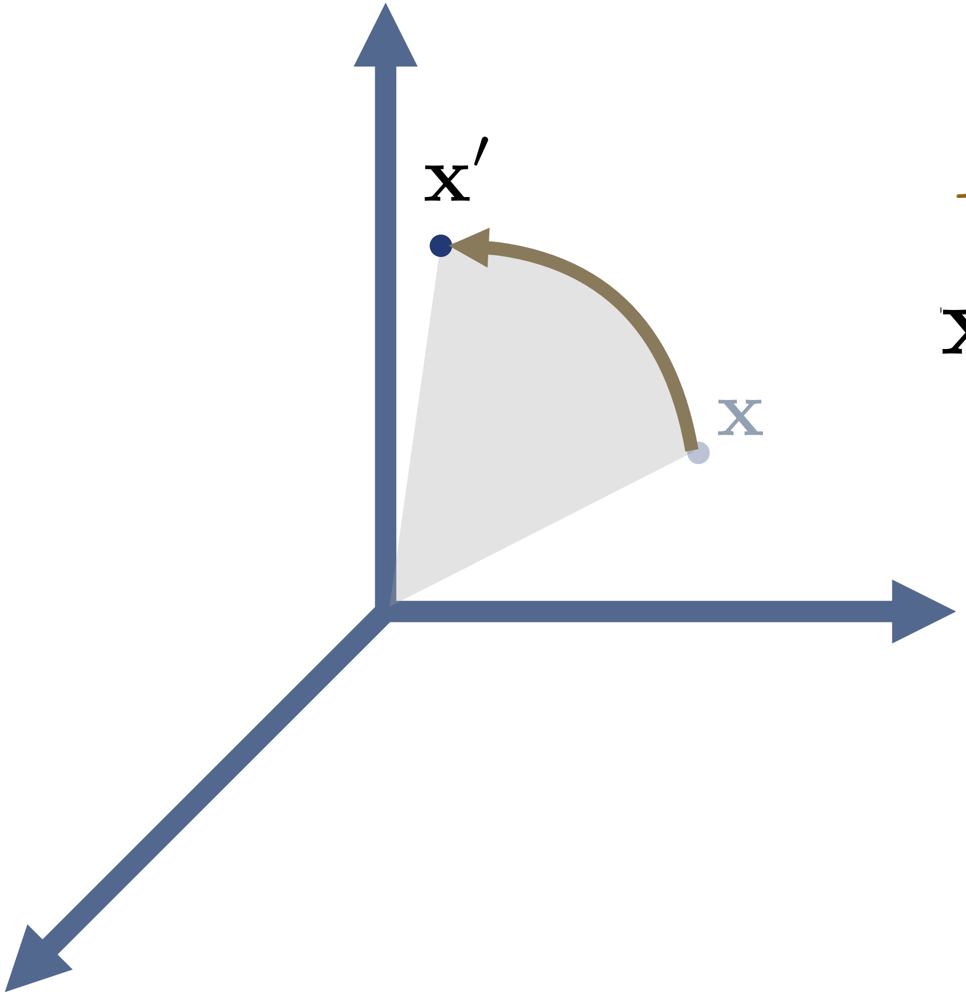
- Translate it
- Rotate it
- Today we'll consider also
- Scale it

$$\mathbf{X}' = \mathbf{X} + \mathbf{t}, \mathbf{t} \in \mathbb{R}^3$$

$$\mathbf{X}'^T = \mathbf{R} \cdot \mathbf{X}^T, \mathbf{R} \in \mathbf{SO}(3)$$

$$\mathbf{X}' = s\mathbf{X}, s \in \mathbb{R}$$

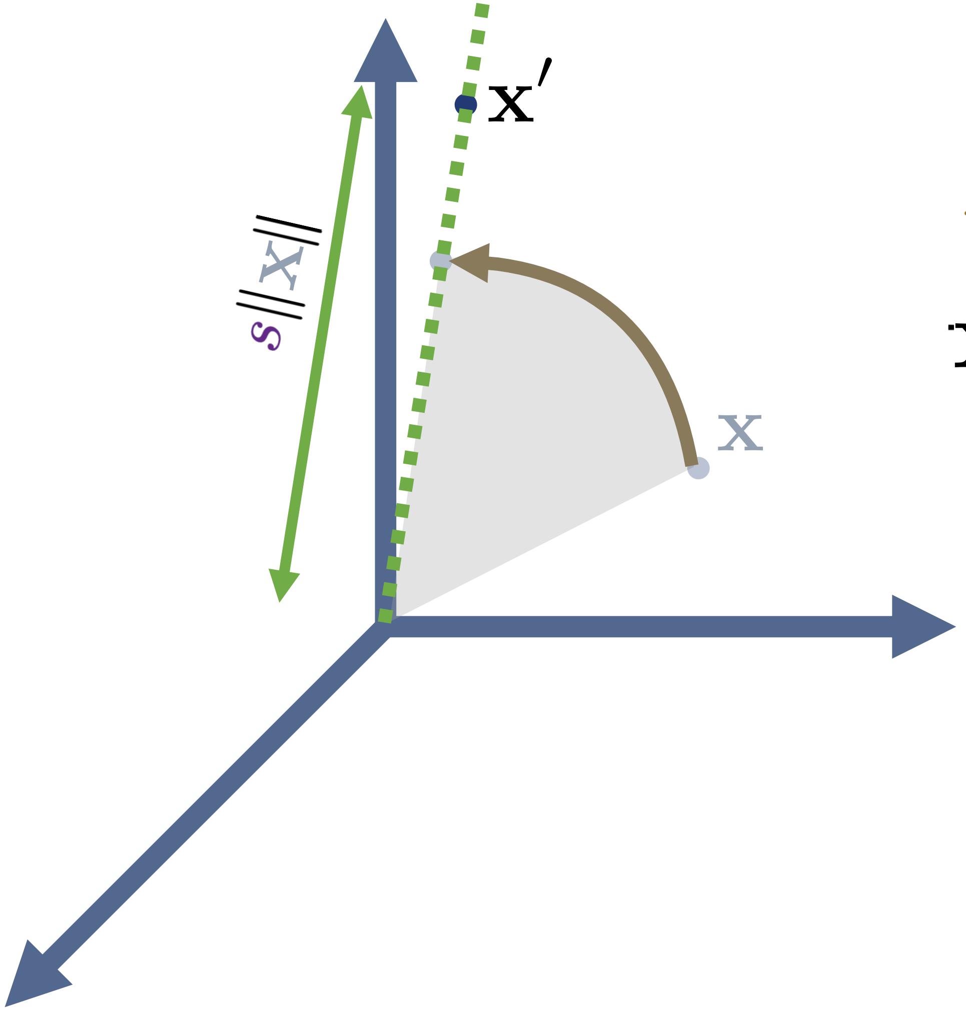
# Rigid transformation + scale




$$R : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\mathbf{x}' = R\mathbf{x}$$

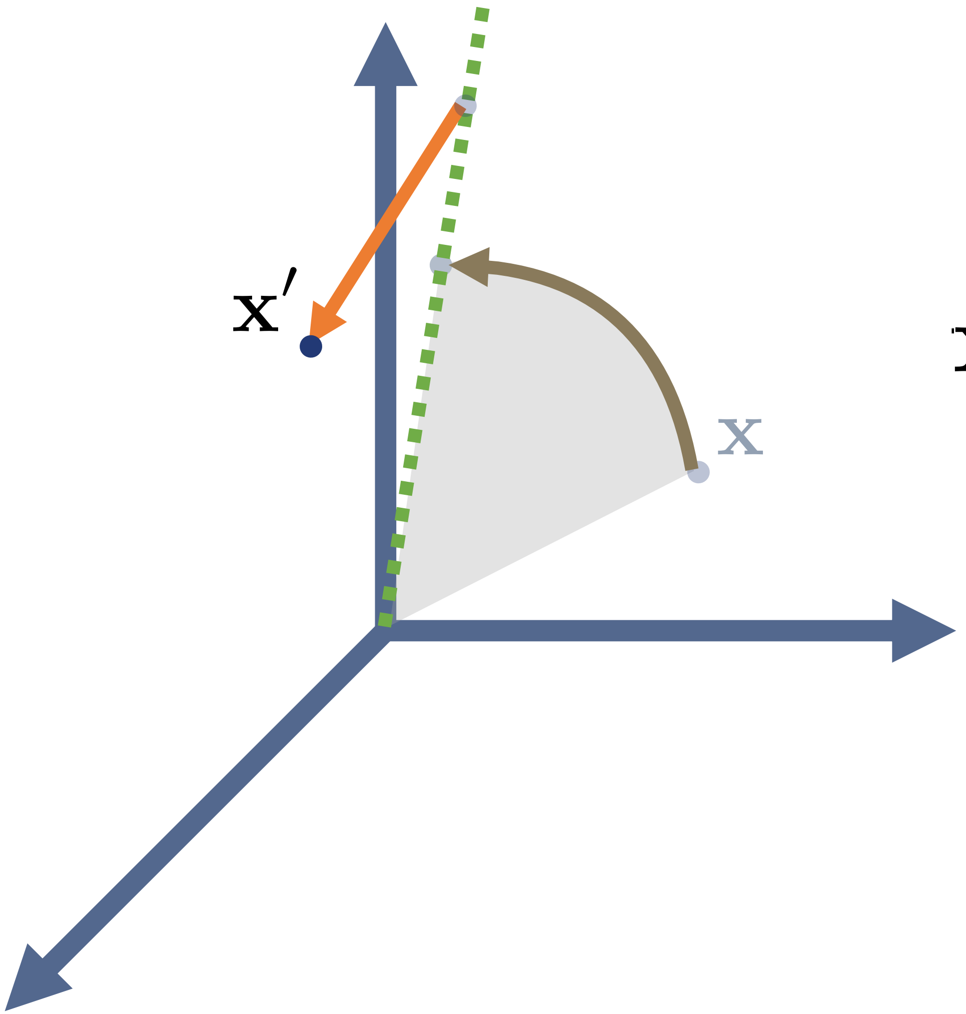
# Rigid transformation + scale




$$R : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\mathbf{x}' = sR\mathbf{x}$$

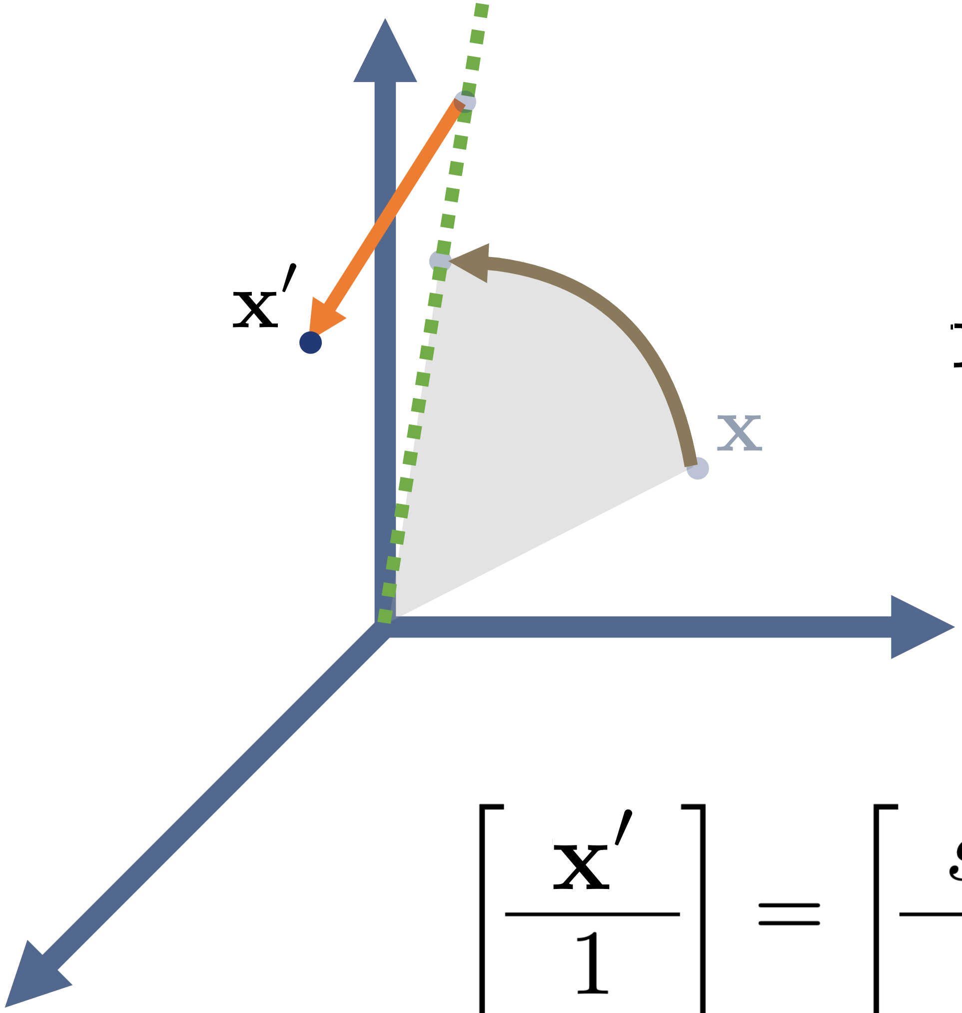
# Rigid transformation + scale




$$R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\mathbf{x}' = sR\mathbf{x} + \mathbf{t}$$

# Rigid transformation + scale



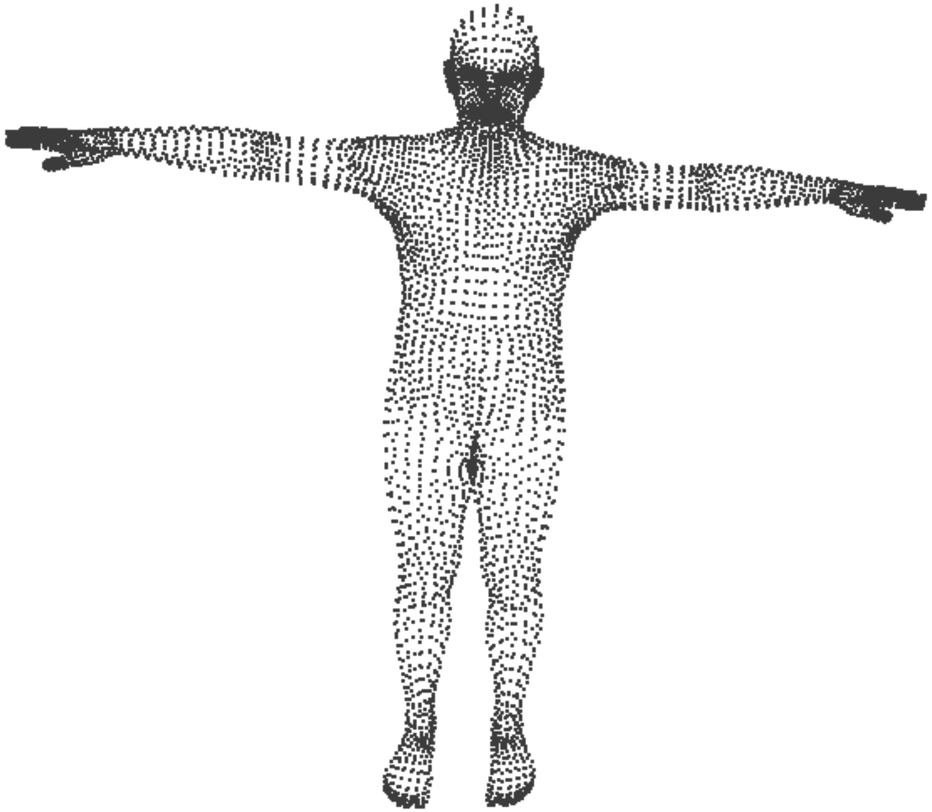
$$R : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\mathbf{x}' = sR\mathbf{x} + \mathbf{t}$$

$$\left[\begin{array}{c}\mathbf{x}' \\ \hline 1\end{array}\right] = \left[\begin{array}{c|c} s\mathbf{R} & \mathbf{t} \\ \hline 0 & 1\end{array}\right] \left[\begin{array}{c}\mathbf{x} \\ \hline 1\end{array}\right]$$

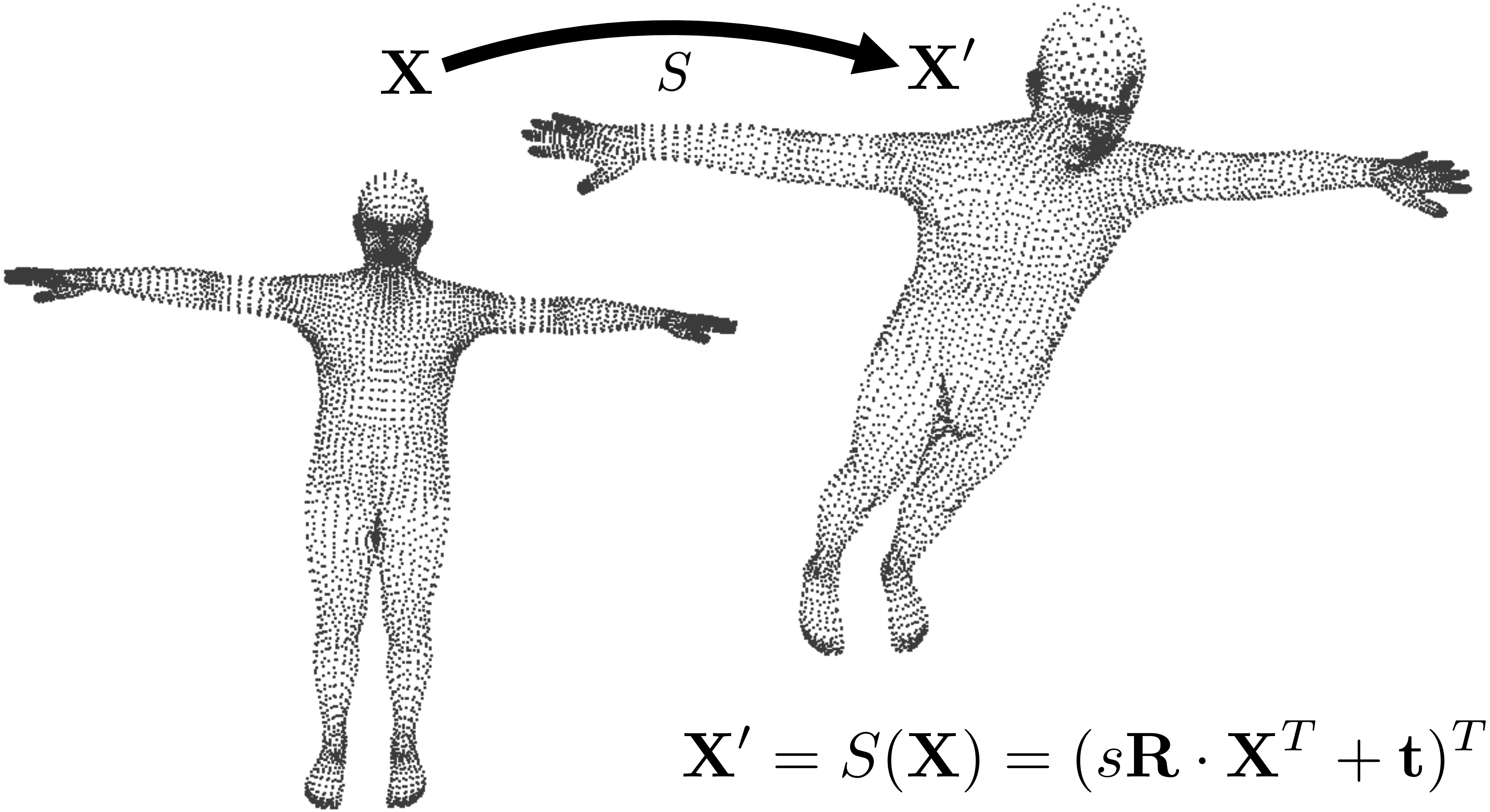
# Procrustes alignment problem definition

$$\mathbf{X} \xrightarrow{S} \mathbf{X}'$$



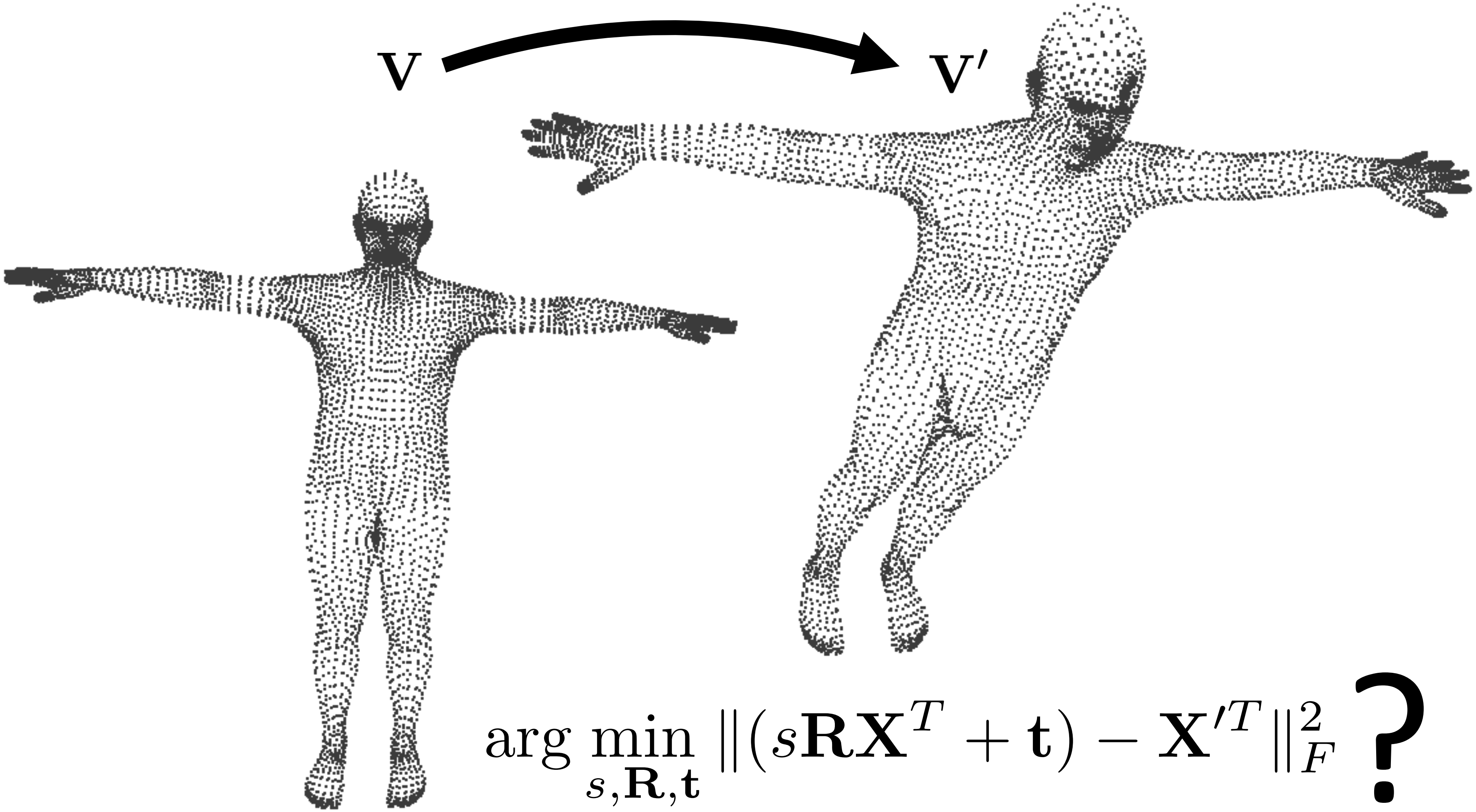
?

# Procrustes alignment problem definition




$$\mathbf{X}' = S(\mathbf{X}) = (s\mathbf{R} \cdot \mathbf{X}^T + \mathbf{t})^T$$

# Procrustes alignment problem definition




$$\arg\min_{s,\mathbf{R},\mathbf{t}} \|(s\mathbf{R}\mathbf{X}^T + \mathbf{t}) - \mathbf{X}'^T\|_F^2$$ ?

# Why Procrustes?

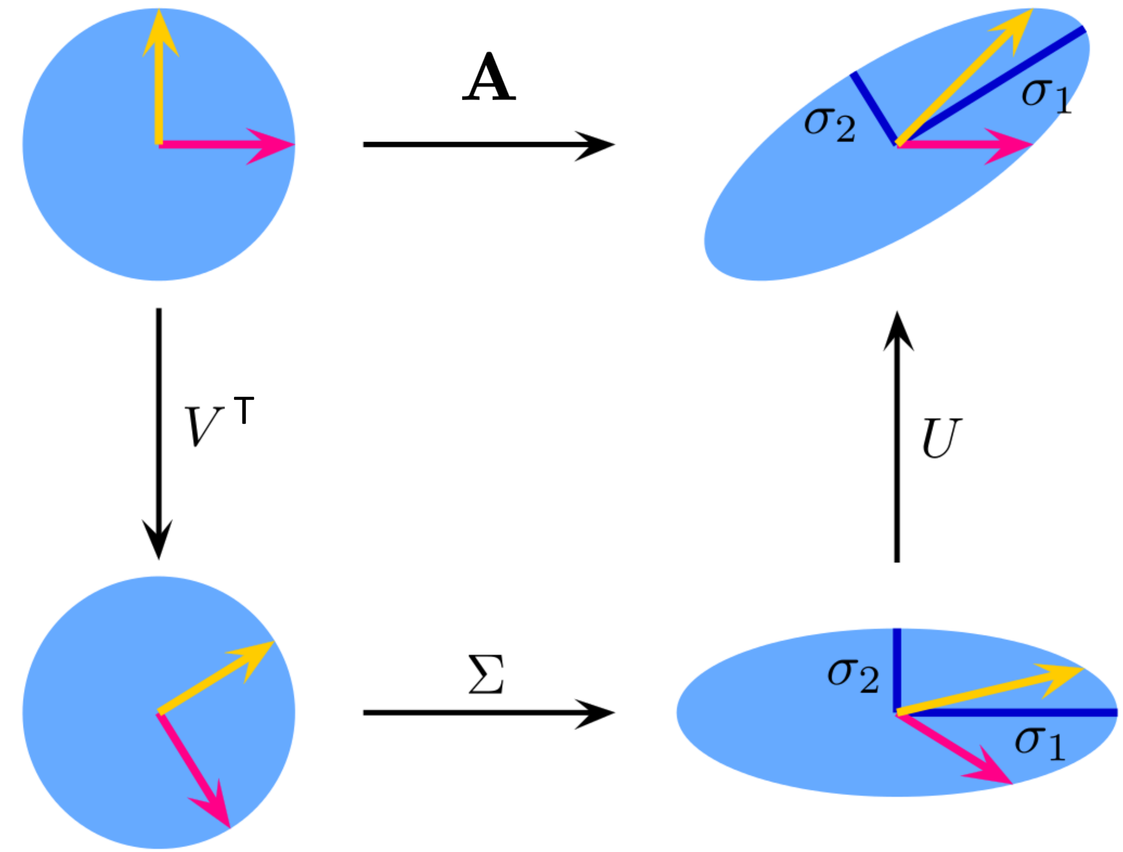
The name *Procrustes* (Greek: Προκρούστης) refers to a bandit from Greek mythology who made his victims fit his bed either by stretching their limbs or cutting them off.

Prcrustes means “he who stretches”

# Optimisation Problem

$$s, \mathbf{R}, \mathbf{t} = \arg\min_{s, \mathbf{R}, \mathbf{t}} \sum_i \| s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i \|^2$$

# Quick recap of SVD




$$A = U\Sigma V^T$$

# Quick recap of SVD

in general, applied to a real matrix:

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathsf{T}}$$

$$\mathbf{A} \equiv (M \times N) \text{ real}$$

$$\mathbf{U} \equiv (M \times M) \text{ orthogonal, unit norm}$$

$$\mathbf{V} \equiv (N \times N) \text{ orthogonal, unit norm}$$

$$\Sigma \equiv (M \times N) \text{ diagonal}$$

warning: this is not the vertex matrix!

* See also clarification slide 1 at the end of slide deck

# Procrustes alignment steps

$\mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_n]^T$ $\mathbf{Y} = [\mathbf{y}_1 \ldots \mathbf{y}_n]^T$ Matrices of points $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{(N\times 3)}$ $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^{(3\times 1)}$

$$\bar{\mathbf{Y}}^T\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^T$$
$$\mathbf{R} = \mathbf{U}\mathbf{V}^T$$

Optimal **rotation** obtained by computing SVD on the point cross-covariance

$$\mathbf{t} = s\mathbf{R}\bar{\mathbf{x}} - \bar{\mathbf{y}}$$

**Translation** is the centroid difference

$$\mathbf{s} = \frac{\mathrm{tr}(\Sigma)}{\|\bar{\mathbf{X}}\|_F^2}$$

**Scale** is a quotient of eigenvalue sums

# Proof

$$s, \mathbf{R}, \mathbf{t} = \arg\min_{s,\mathbf{R},\mathbf{t}} \sum_i \|s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i\|^2$$

# Procrustes derivation: translation

$$\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{(N \times 3)}$$

$$\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^{(3 \times 1)}$$

$$s, \mathbf{R}, \mathbf{t} = \arg\min_{s, \mathbf{R}, \mathbf{t}} E$$

$$E \equiv \sum_i \|s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i\|^2$$

Minimize the L2 distance between transformed source points and target points

$$= \sum_i (s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i)^\mathsf{T} (s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i)$$

$$= \sum_i s^2 \mathbf{x}_i^\mathsf{T} \mathbf{x}_i + \mathbf{t}^\mathsf{T}\mathbf{t} + \mathbf{y}_i^\mathsf{T}\mathbf{y}_i + 2s\mathbf{x}_i^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{t} - 2s\mathbf{x}_i^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{y}_i - 2\mathbf{t}^\mathsf{T}\mathbf{y}_i$$

# Procrustes derivation: translation

We remove the elements that do not depend on the translation and solve for t.

$$\mathbf{t} = \arg\min_{\mathbf{t}} E = \arg\min_{\mathbf{t}} \sum_i \cancel{s^2\mathbf{x}_i^\mathsf{T}\mathbf{x}_i} + \mathbf{t}^\mathsf{T}\mathbf{t} + \cancel{\mathbf{y}_i^\mathsf{T}\mathbf{y}_i} + 2s\mathbf{x}_i^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{t} - \cancel{2s\mathbf{x}_i^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{y}_i} - 2\mathbf{t}^\mathsf{T}\mathbf{y}_i$$

$$= \arg\min_{\mathbf{t}} \sum_i \mathbf{t}^\mathsf{T}\mathbf{t} + 2s\mathbf{x}_i^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{t} - 2\mathbf{t}^\mathsf{T}\mathbf{y}_i$$

$$\bar{\mathbf{x}} \equiv \frac{\sum_i \mathbf{x}_i}{N}, \bar{\mathbf{y}} \equiv \frac{\sum_i \mathbf{y}_i}{N}$$

Compute the centroid of the point clouds

$$\mathbf{t} = \arg\min_{\mathbf{t}} E = \arg\min_{\mathbf{t}} (\mathbf{t}^\mathsf{T}(2s\mathbf{R}\bar{\mathbf{x}} + \mathbf{t} - 2\bar{\mathbf{y}})) = \bar{\mathbf{y}} - s\mathbf{R}\bar{\mathbf{x}}$$

So given s and R, we can compute the translation t

# Procrustes derivation: Rotation

Subtract the centroid from the points to obtain a simpler expression for E

$$\bar{\mathbf{x}}_i \equiv \mathbf{x}_i - \bar{\mathbf{x}}, \bar{\mathbf{y}}_i \equiv \mathbf{y}_i - \bar{\mathbf{y}}$$

subject to $\mathbf{R} \in SO(3)$

$$E = \sum_i \|s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i\|^2 = \sum_i \|s\mathbf{R}\bar{\mathbf{x}}_i - \bar{\mathbf{y}}_i\|^2$$

Optimal rotation does not depend on scale

$$\mathbf{R} = \arg\min_{\mathbf{R}} ||\mathbf{R}\bar{\mathbf{X}}^T - \bar{\mathbf{Y}}^T||_F^2$$

# Procrustes derivation: Rotation

Subtract the centroid from the points to obtain a simpler expression for E

$$\bar{\mathbf{x}}_i \equiv \mathbf{x}_i - \bar{\mathbf{x}}, \bar{\mathbf{y}}_i \equiv \mathbf{y}_i - \bar{\mathbf{y}}$$

$$E = \sum_i \|s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i\|^2 = \sum_i \|s\mathbf{R}\bar{\mathbf{x}}_i - \bar{\mathbf{y}}_i\|^2$$

Optimal rotation does not depend on scale

$$\begin{aligned}\mathbf{R} &= \arg\min_{\mathbf{R}} ||\mathbf{R}\bar{\mathbf{X}}^T - \bar{\mathbf{Y}}^T||_F^2 \\ &= \arg\min_{\mathbf{R}} \langle \mathbf{R}\bar{\mathbf{X}}^T - \bar{\mathbf{Y}}^T, \mathbf{R}\bar{\mathbf{X}}^T - \bar{\mathbf{Y}}^T \rangle_F \\ &= \arg\min_{\mathbf{R}} \|\mathbf{R}\bar{\mathbf{X}}^T\|_F^2 + \|\bar{\mathbf{Y}}^T\|_F^2 - 2\langle \mathbf{R}\bar{\mathbf{X}}^T, \bar{\mathbf{Y}}^T \rangle_F \\ &= \arg\min_{\mathbf{R}} \|\bar{\mathbf{X}}^T\|_F^2 + \|\bar{\mathbf{Y}}^T\|_F^2 - 2\langle \mathbf{R}\bar{\mathbf{X}}^T, \bar{\mathbf{Y}}^T \rangle_F \\ &= \arg\max_{\mathbf{R}} \langle \mathbf{R}, \bar{\mathbf{Y}}^T\bar{\mathbf{X}} \rangle_F\end{aligned}$$

Rotation does not change norm!

Inner product should be maximum!

# Procrustes derivation: Rotation

Subtract the centroid from the points to obtain a simpler expression for E

$$\bar{\mathbf{x}}_i \equiv \mathbf{x}_i - \bar{\mathbf{x}}, \bar{\mathbf{y}}_i \equiv \mathbf{y}_i - \bar{\mathbf{y}}$$

$$E = \sum_i \|s\mathbf{R}\mathbf{x}_i + \mathbf{t} - \mathbf{y}_i\|^2 = \sum_i \|s\mathbf{R}\bar{\mathbf{x}}_i - \bar{\mathbf{y}}_i\|^2$$

Optimal rotation does not depend on scale

$$\begin{aligned}
\mathbf{R} &= \arg\min_{\mathbf{R}} ||\mathbf{R}\bar{\mathbf{X}}^T - \bar{\mathbf{Y}}^T\|_F^2 \\
&= \arg\min_{\mathbf{R}} \langle \mathbf{R}\bar{\mathbf{X}}^T - \bar{\mathbf{Y}}^T, \mathbf{R}\bar{\mathbf{X}}^T - \bar{\mathbf{Y}}^T \rangle_F \\
&= \arg\min_{\mathbf{R}} \|\mathbf{R}\bar{\mathbf{X}}^T\|_F^2 + \|\bar{\mathbf{Y}}^T\|_F^2 - 2\langle \mathbf{R}\bar{\mathbf{X}}^T, \bar{\mathbf{Y}}^T \rangle_F \\
&= \arg\min_{\mathbf{R}} \|\bar{\mathbf{X}}^T\|_F^2 + \|\bar{\mathbf{Y}}^T\|_F^2 - 2\langle \mathbf{R}\bar{\mathbf{X}}^T, \bar{\mathbf{Y}}^T \rangle_F \\
&= \arg\max_{\mathbf{R}} \langle \mathbf{R}, \bar{\mathbf{Y}}^T\bar{\mathbf{X}} \rangle_F \\
&= \arg\max_{\mathbf{R}} \langle \mathbf{R}, U\Sigma V^T \rangle_F \\
&= \arg\max_{\mathbf{R}} \langle U^T\mathbf{R}V, \Sigma \rangle_F \\
&= \arg\max_{\mathbf{R}} \langle S, \Sigma \rangle_F \quad \text{where } S = U^T\mathbf{R}V
\end{aligned}$$

Rotation does not change norm!

Inner product should be maximum!

Cross-covariance matrix

SVD decomposition

# Procrustes derivation: Rotation

$$= \arg\max_{\mathbf{R}} \langle S, \Sigma \rangle_F \quad \text{where } S = U^T \mathbf{R} V$$

What kind of matrix is $S$ ?

What kind of matrix is $\Sigma$?

# Procrustes derivation: Rotation

$$= \arg\max_{\mathbf{R}} \langle S, \Sigma \rangle_F \quad \text{where } S = U^T \mathbf{R} V$$

What kind of matrix is $S$ ? Orthogonal

What kind of matrix is $\Sigma$? Diagonal

Hence the quantity above is maximised when $S$ equals the identity, hence:

$$I = U^T \mathbf{R} V$$

$$\mathbf{R} = U V^T$$

SVD of cross-covariance of pointsclouds

$$\bar{\mathbf{Y}}^T \mathbf{X} = U \Sigma V^T$$

# Procrustes derivation: scale

Optimize **scale** given the rotation

$$s = \arg\min_s E = \arg\min_s \left(\sum_i s^2 \bar{\mathbf{x}}_i^\mathsf{T} \bar{\mathbf{x}}_i + \cancel{\bar{\mathbf{y}}_i^\mathsf{T} \bar{\mathbf{y}}_i} - 2s\bar{\mathbf{x}}_i^\mathsf{T} \mathbf{R}^\mathsf{T} \bar{\mathbf{y}}_i\right)$$

$$= \arg\min_s \left(s^2 \sum_i (\bar{\mathbf{x}}_i^\mathsf{T} \bar{\mathbf{x}}_i) - 2s \sum_i (\bar{\mathbf{x}}_i^\mathsf{T} \mathbf{R}^\mathsf{T} \bar{\mathbf{y}}_i)\right)$$

$$= \arg\min_s (s^2 a - 2sb) = \frac{b}{a} = \frac{\sum_i (\bar{\mathbf{x}}_i^\mathsf{T} \mathbf{R}^\mathsf{T} \bar{\mathbf{y}}_i))}{\sum_i (\bar{\mathbf{x}}_i^\mathsf{T} \bar{\mathbf{x}}_i)}$$

$$= \frac{\mathrm{tr}(\bar{\mathbf{X}} \mathbf{R}^T \bar{\mathbf{Y}}^T)}{\|\bar{\mathbf{X}}\|_F^2} = \frac{\mathrm{tr}(\mathbf{R}^T \bar{\mathbf{Y}}^T \bar{\mathbf{X}})}{\|\bar{\mathbf{X}}\|_F^2} = \frac{\mathrm{tr}(\mathbf{V}\mathbf{U}^T \mathbf{U}\Sigma\mathbf{V}^T)}{\|\bar{\mathbf{X}}\|_F^2} = \frac{\mathrm{tr}(\Sigma)}{\|\bar{\mathbf{X}}\|_F^2}$$

We used:
1) Trace invariance with shifts
2) $\bar{\mathbf{Y}}^T \bar{\mathbf{X}} = \mathbf{U}\Sigma\mathbf{V}^T, \; \mathbf{R} = \mathbf{U}\mathbf{V}^T$
3) Trace equals sum of eigenvals for square matrices

# Clarification slides
# (not included in the video recordings)

# Clarification slide 1 (not in the video)

$$\bar{\mathbf{X}} = [\mathbf{x}_1 - \bar{\mathbf{x}}, \ldots, \mathbf{x}_n - \bar{\mathbf{x}}]^T$$

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{i}^{N} \mathbf{x}_i$$

Normalized pointcloud. The centroid is subtracted to all points to center the pointcloud at the origin

Centroid